Homework 8

Exercises to Turn In. Due Date: Friday, March 25.

1. Ross 4.23

2. Ross 4.27. Hint: Read Example 1.9(A).

The next three problems are adapted from Resnick's Adventures in Stochastic Processes text:

3. The Moran model in genetics has individuals being replaced one at a time. Allele frequencies in the population thus move by at most one part. Specifically, the Markov chain describing the allele frequencies is given by

$$P_{ij} = \begin{cases} \frac{i(M-i)}{M^2} & \text{if } |i-j| = 1, 0 < i < M \\ 1 - \frac{2i(M-i)}{M^2} & \text{if } i = j, 0 < i < M \\ 1 & \text{if } i = j = 0 \text{ or } i = j = M \\ 0 & \text{otherwise} \end{cases}$$

Find the probability $u_{i,M}$ that the allele is fixed in the population of size M (being fixed means absorption at state M.) Hint: Find a recurrence relation for $u_i = u_{i,M}$.

4. The Wright model in genetics has all individuals being replaced as a group with binomial replacements. Allele frequencies in the population thus can move by all possible amounts. Specifically, the Markov chain describing the allele frequencies is given by

$$P_{ij} = \binom{M}{j} \left(\frac{i}{M}\right)^j \left(1 - \frac{i}{M}\right)^{M-j}$$

Show that the probability the allele is fixed in a population of size M that follows the Wright model for binomial generational replacement is the same as the probability you found for the Moran model where individuals are replaced one at a time. Hint: Show the solution in the previous problem solves the recurrence relation for the fixation probabilities for the Wright model.

5. Three cards are placed in a row in one of 6 possible orders. The order is changed at each step by the following procedure: pick the card to the leftmost with probability p or the card to the rightmost with probability $(1-p)$. Place the chosen card in the middle of the other two. What is the mean number of choices (moves) required to return the cards to their original order? Hints: (a) If arranged properly, this could look a lot like the process in Problem 2. (b) Since the chain is smallish (6 states in total), one can use the brute force technique shown in class to get the answer. Using Maple or Mathematica might help. (c) But then one discovers that the answer is very, very nice. For instance, it does not depend on p. That suggests that there is an elegant approach (for instance, by using recursions in an elegant way) that gives the answer as well.