Homework 1
Routine Practice Exercises Not to Be Turned In

1. What is the median of an exponential distribution with rate λ?

2. Let X and Y have joint density function

$$f(x,y) = \begin{cases} cxy & \text{if } x > 0, y > 0, x + y < 2 \\ 0 & \text{otherwise} \end{cases}$$

   What is c? What is EX? What is EXY?

3. Calculate the moment generating function for a Binomial distribution. Use the moment generating function to find the mean and the variance of the binomial distribution. Use the moment generating function to prove that the sum of two Binomial distributions with the same probability parameter, p, is also a Binomial distribution. Use the moment generating function to show that the limit as $np \to \lambda$ when $n \to \infty$ the Binomial distribution tends to a Poisson distribution with rate λ.

4. Suppose there are but two eye colors in the world: green and brown. Suppose eye color is controlled by one gene. Green is recessive meaning that a person's eyes are green if and only if they inherited a green gene from both their mother and their father. Brown is dominant meaning that if a person has brown eyes they could have one green gene and one brown gene. Now suppose a green eyed mother has three children (no twins) with a brown eyed father whose own father had green eyes. (This means that the mother has 2 green genes and the father has one green gene and one brown gene.) What is the probability that a given child has green eyes? What is the probability that the oldest child and the middle child both have green eyes? What is the probability the middle child and the youngest child both have green eyes? What is the probability the oldest child and the youngest child both have green eyes? What is the probability all three children have green eyes? What does this say about the events: child i has green eyes?

5. Ross: 1.2. Let X be a continuous random variable with distribution function $F(x)$. Show that $F(X)$ is uniform on $(0,1)$ and that if U is uniform on $(0,1)$, then $F^{-1}(U)$ is a random variable with distribution function F. Here F^{-1} is the functional inverse.

Exercises to Turn In. Due Date: Friday, January 21

1. Ross: 1.11 (a)-(c). (Note the double use of "P" in the following problem. Its meaning is clear in each instance.) If X is a non-negative integer valued random variable, then the probability generating function is defined for $|z| \leq 1$ by

$$P(z) = E(z^X) = \sum_{j=0}^{\infty} z^j P\{X = j\}$$

   a) Show that $\frac{d^k}{dz^k} P(z)|_{z=0} = k! P\{X = k\}$

   b) With 0 being considered even, show that $P\{X \text{ is even}\} = \frac{P(-1) + P(1)}{2}$

   c) If X is Binomial with parameters p and n, show that $P\{X \text{ is even}\} = \frac{1 + (1 - 2p)^n}{2}$

2. Let $P(z)$ be the probability generating function as in the previous problem. Let $q_k = P\{X > k\}$ and define $Q(z) = \sum_{k=0}^{\infty} q_k z^k$. Show that $Q(z) = \frac{1 - P(z)}{1 - z}$ for $0 \le z < 1$. How do you compute $P\{X > k\}$ using the generating function $Q(z)$? What does this mean about computing $P\{X > k\}$ using the probability generating function $P(z)$?

3. Ross: 1.1. Let N denote a non-negative integer valued random variable. Show that

$$EN = \sum_{k=1}^{\infty} P\{N \ge k\} = \sum_{k=0}^{\infty} P\{N > k\}.$$

In general, show that if X is non-negative with distribution F then,

$$EX = \int_0^{\infty} (1 - F(x))dx$$

and

$$EX^n = \int_0^{\infty} nx^{n-1}(1 - F(x)).$$

4. Consider an infinite sequence of fair coin tosses with outcome ω. Let E_n be the event that there are at least $2 \log_2 n$ heads in a row starting at coin toss n. Bound the probability of E_n and use that bound to show that $P\{E_n \text{ i.o.}\} = 0$.

5. Consider an infinite sequence of fair coin tosses with outcome ω. Let E_n be the event that there is a run of length 1 starting at coin toss n. That is, coin toss n is heads and coin toss $n + 1$ is tails. Show that $P\{E_n \text{ i.o.}\} = 1$. Note that $\{E_n : n \ge 1\}$ are not independent. What is a subset of events that are independent?

From Çinlar: Introduction to Stochastic Processes

Then said unto him, Say now Shibboleth: and he said Sibboleth: for he could not frame to pronounce it right. Then they took him, and slew him.
The Bible, The Book of Judges 12:6

Failure to distinguish between an outcome and an event is unlikely to have as sever a consequence as the fate which befell the warrior challenged by the men of ancient Gilead. Yet, probabilistic reasoning is a part of our culture today, and an appreciation of it cannot be acquired without developing a precise feeling for its essential ingredients.

It is important to remember always that an event is a collection of outcomes, while a random variable is a function. A random variables assigns a value to each outcome; a probability measure assigns a value to each event. One talks of the probability of an event, never of the probability of an outcome. A certain amount of confusion is caused by the historical mistakes made while the subject was developing, and the insistence of certain teachers on repeating them.

The present axiomatic foundations of the theory were laid by Kolmogorov in 1933. Since then, the progress in probability theore has been very rapid. This progression was especially aided by the discovery of unsuspected applications to pure mathematics on the one hand, and by an ever increasing demand from other scientists and engineers on the other.