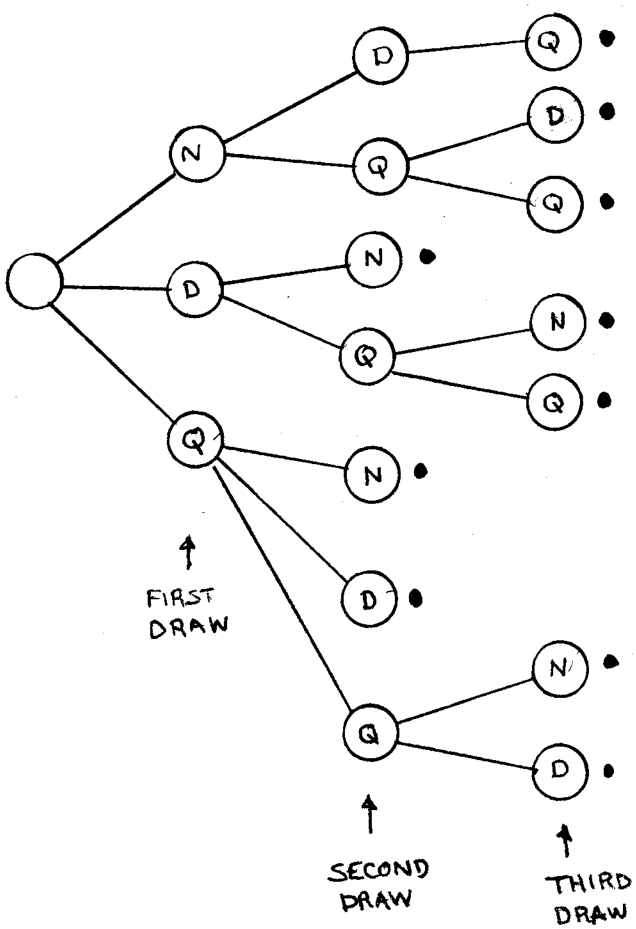
1) A jar contains one nickel, one dime, and 2 quarters. Two coins are drawn out, one after the other, without replacement. The values of the coins are recorded in the order that they are drawn out. If the value of the second coin drawn out is EQUAL TO OR HIGHER than that of the first coin drawn out, then (and only then) a third coin is drawn out, and its value is noted. How many elements are in the corresponding sample space? Examples: one element might be NDQ - draw a nickel, then a dime, then a quarter. Another might be QD.

a) 12 b) 11 c) 13

d) 10 e) 9 f) none of the above

Draw a tree:



The process terminates when a dot • is reached.

Ten dots.

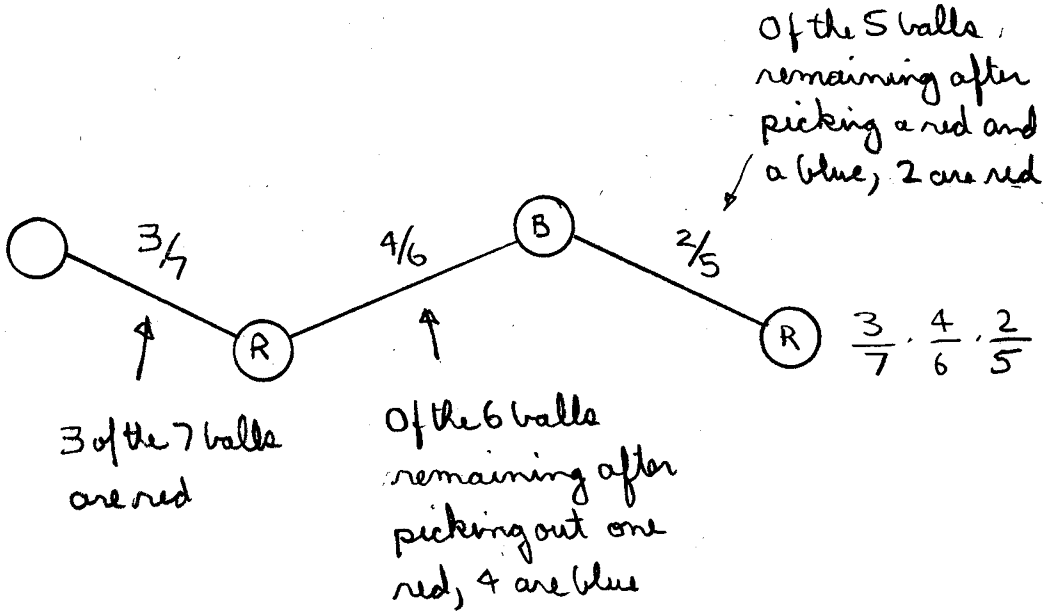
Answer: 10

2) An urn contains 4 blue balls and 3 red balls. Three balls are drawn out, one after the other, without replacement. What is the probability of getting a red, then a blue, then a red?

a) 36/343 b) 24/343 c) 4/35

d) 1/8 e) 12/35 f) none of the above

<u>One</u> branch of the tree for this process looks like:



This is the only branch that corresponds to a red, then a blue, then a red.

Answer:

$$\frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} = \underline{\underline{\frac{4}{35}}}$$

3) Two of the cokes in a six pack of cokes are contaminated with ammonia. You select, at random, two of the six cokes and drink them. What is the probability that you have consumed a coke (one or more) containing ammonia.

a) 3/5 b) 11/15 c) 8/15

d) 3/10 e) 2/3 f) none of the above

Pr (1 or more contaminated) = 1 − Pr (0 contaminated)

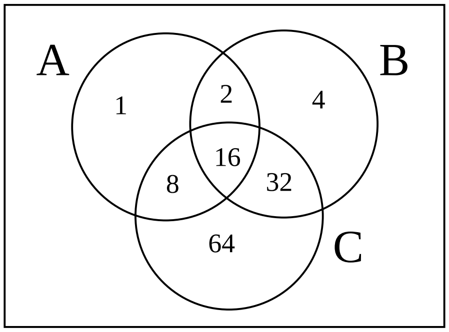
$$= 1 - \frac{\binom{4}{2}}{\binom{6}{2}}$$

$\binom{4}{2}$ ← # ways to choose 2 of the 4 that are not contaminated.

$\binom{6}{2}$ ← # ways to choose 2 of the 6 cokes.

$$= 1 - \frac{6}{6 \cdot 5/2} = 1 - \frac{6}{15} = 1 - 2/5 = \underline{\underline{3/5}}.$$

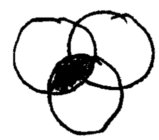
4) Consider the following Venn diagram for sets A, B, C. Shown in this diagram are the number of elements in each indicated subset. Let $F = B \cap ((A \cap C)')$. Find $n(F)$, the number of elements in F.



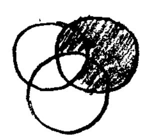
a) 20 b) 58 c) 38

d) 34 e) 54 f) none of the above

$F = B \cap ((A \cap C)')$. This has to be a subset of B (since $(B \cap \text{anything}) \subset B$). The question is, what does $(A \cap C)'$ "cover up" in B. Note $(A \cap C)'$ = everything outside $A \cap C$. But $A \cap C$ corresponds to



Everything outside $A \cap C$ but also in B is:



Answer: $32 + 4 + 2 = 38$

5) On a trip, Madeleine Albright plans to visit each of the cities Athens, Berlin, Cairo, Dusseldorf, Edinburgh, and Frankfort. How many different itineraries are possible provided that she visits Berlin and Frankfort after visiting all of the other cities. (Example: She visits Cairo then Athens then Edinburgh then Dusseldorf then Frankfort then Berlin.)

a) 24 b) 96 c) 720

d) 48 e) 360 f) none of the above

$\underbrace{ACDE}$ BF

These four are visited first

4! ways to do this – 4 ways to choose the 1^{st} city,
3 ways to choose the 2^{nd} city,
2 ...
1 ...

or $P(4,4) = 4!$

ABCD $\underbrace{BF}$

These are visited last. 2 ways to do this → FB or BF.

$P(2,2) = 2! = 2$

Answer: $4! \cdot 2 = 48$.

6) Three cars are to be parked in 8 parking slots. How many ways can can this be done? Example: One way is to park car 1 in slot 6, car 2 in slot 5, and car 3 in slot 1. All that matters is the information as to which car is in which slot.

a) 256 b) 512 c) 56

d) 336 e) 21 f) none of the above

Car 1 can be parked in any of the 8 slots: 8 ways to do this
Car 2 can be parked in any of the 7 remaining slots: 7 ways
Car 3 can be parked in any of the 6 remaining slots: 6 ways

Answer: $8 \cdot 7 \cdot 6 = 336$

Equivalently $P(8,3) = 336$

Choose 3 slots keeping track of order (using car#).

7) Suppose A and B are subsets of a universal set U. Suppose that $Pr[A' \cap B'] = .22$ and $Pr[A] = .32$ and $Pr[B] = .72$. Find $Pr[A \cap B]$.

a) .26 b) .54 c) .46

d) .06 e) .52 f) none of the above

$$\underbrace{Pr(A' \cap B') = Pr((A \cup B)')}_{\text{deMorgan's laws}} = .22 = 1 - Pr(A \cup B)$$

since $Pr(E') = 1 - Pr(E)$ for all $E \subset U$.

So $Pr(A \cup B) = 1 - .22 = .78$.

Now use

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

$$\Rightarrow \quad .78 = .32 + .72 - Pr(A \cap B)$$

So $.78 - .32 - .72 = -.26 = -Pr(A \cap B)$

$$\Rightarrow \quad Pr(A \cap B) = .26$$

8) The probability of rolling a one on an unfair die is 1/10. The probability of rolling a 2 is 1/10. The probability of rolling a 3 is 1/5, of rolling a 4 is 1/5, of rolling a 5 is 1/5, of rolling a 6 is 1/5. Suppose the die is rolled twice. What is the probability that the sum of the two rolls is 4?

a) 1/2 b) 1/18 c) 3/50

d) 1/12 e) 1/20 f) none of the above

a "4" can be obtained by rolling a 1 and a 3 or
2 and a 2 or
3 and a 1.

ONE ON 1ST ROLL THREE ON 2nd ROLL

$Pr(1 \text{ and a } 3) = Pr(1 \text{ on 1st Roll})\ Pr(3 \text{ on 2nd Roll})$

INDEPENDENT EVENTS

$= 1/10 \cdot 1/5 = \frac{1}{50}$

$Pr(3 \text{ and a } 1) = \cdots$ (SAME REASONING) $= \frac{1}{50}$

$Pr(2 \text{ and a } 2) = Pr(2 \text{ on 1st Roll})\ Pr(2 \text{ on 2nd Roll})$

$= 1/10 \cdot 1/10 = 1/100$

answer: $1/50 + 1/50 + 1/100 = 5/100 = \underline{\underline{1/20}}$

9) Alicia will take a trip consisting of 3 segments in South America. The schedules show that:

1) Airlines A, B, and C all fly segment 1.
2) Airlines A, B, C, D all fly segment 2.
3) Airlines A, C, D, E all fly segment 3.

A trip package consists of picking one airline for each segment. How many possible trip packages* can she choose from, where she does NOT take the same airline for all 3 segments?

*Examples: One possible trip package is ABD (take A then B then D). Another is BCC.

a) 45 b) 44 c) 48

d) 17 e) 46 f) none of the above

With no restrictions, there are

$$\underbrace{3}_{\text{\# choices for leg 1}} \cdot \underbrace{4}_{\text{\# choices for leg 2}} \cdot \underbrace{4}_{\text{\# choices for leg 3}} = 48 \text{ possible trip packages.}$$

However, 2 of those are AAA and CCC which is ruled out by the statement of the problem.

Answer: $48 - 2 = \underline{\underline{46}}$

10) Ten people line up to go to the theater - five boys and five girls. How many ways can they line up so that the five girls are first in line and the five boys are last in line?

a) C(10,5)/2 b) $5! \cdot 5!$ c) $10!$

d) $2 \cdot 5! \cdot 5!$ e) $10!/2$ f) none of the above

$\underbrace{G_1\, G_2\, G_3\, G_4\, G_5}$ $B_1\, B_2\, B_3\, B_4\, B_5$ Here is one line up as asked for in the problem.

There are 5! ways to permute the girls. This will still leave the 5 girls 1st in line.

There are 5! ways to permute the boys which also leaves the 5 girls 1st in line.

Answer: $5! \cdot 5!$

11) The table shown below gives the values of a random variable X and the density function of X. Unknown are the value of z and the value of p_1. Suppose that $E[X] = 0.4$. Find the value of z.

Value of X	Probability
-5	.10
-1	p_1
0	.20
z	.25
5	.10

a) 3 b) 2 c) 4

d) 2.4 e) 2.6 f) none of the above

Step I: All of the probabilities must add up to 1. So p_1 must be .35.

Step II:

$E(X) = .4$

$$= \underbrace{(-5)\cdot(.10)}+(-1)\cdot(.35) + \overbrace{0\cdot(.20)}^{=0} + z(.25) + \underbrace{5(.10)}$$

cancel

$$= -.35 + z\cdot(.25)$$

ADD .35 TO BOTH SIDES

$$\Rightarrow \quad .75 = z\,(.25)$$

$$\Rightarrow \quad z = \underline{\underline{3}}$$

12) An unfair coin has a probability of 2/3 of landing heads. The coin is flipped 3 times. Then a <u>fair</u> coin is flipped once. What is the probability that one or more of the four tosses landed tails?

a) 16/27 b) 53/54 c) 103/108

d) 3/4 e) 23/27 f) none of the above

$$Pr(\underbrace{1 \text{ or more Tails}}) = 1 - Pr(\underbrace{\text{No Tails}})$$

COMPLEMENTARY EVENTS

$Pr(A') = 1 - Pr(A)$

$Pr(\text{No Tails}) = Pr(\text{all Heads})$

To calculate this draw a tree (or realize the outcomes of the tosses are independent).

$$H \xrightarrow{2/3} H \xrightarrow{2/3} H \xrightarrow{2/3} H \xrightarrow{1/2} H \quad 2/3 \cdot 2/3 \cdot 2/3 \cdot 1/2$$

ONE BRANCH OF TREE

$$Pr(\text{all Heads}) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{4}{27}$$

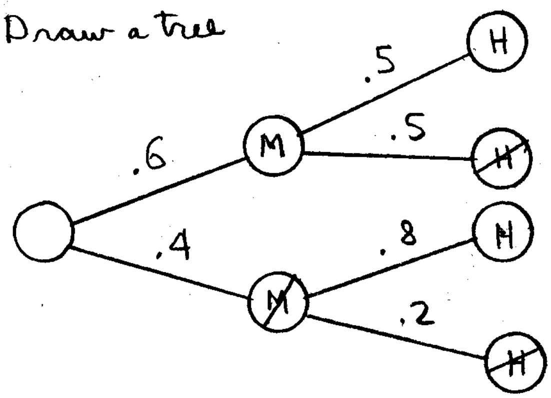
Answer: $1 - 4/27 = \underline{\underline{23/27}}$

13) 60% of the adults in Spain support the monarchy (and 40% don't). Of those adults that support the monarchy, 50% graduated from high school. Of those adults that don't support the monarchy, 80% graduated from high school. What percentage of the adult population of Spain graduated from high school?

a) 70% b) 65% c) 68%

d) 60% e) 62% f) none of the above

Draw a tree



$.6 \cdot .5 = .30$ ← HS graduates that support M

$.4 \cdot .8 = .32$ HS graduates that don't support M

$.62$ ← all HS graduates.

M represents supporting monarchy
H represents graduating from high school
/ represents complementary event.

Answer: 62%

14) Suppose A,B, and C are sets with:

$n(A) = 100$ $n(B) = 80$ $n(C) = 70$

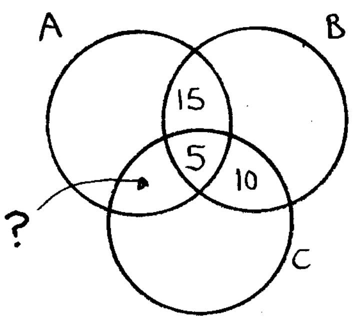
$n(A \cap B) = 20$ $n(B \cap C) = 15$

$n(A \cap B \cap C) = 5$ $n(A \cup B \cup C) = 190$

What is $n(A \cap C)$?

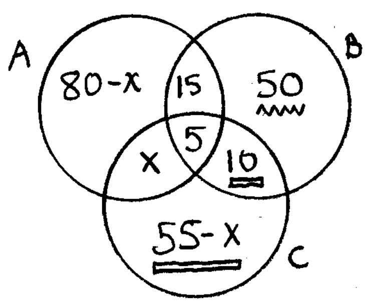
a) 45 b) 30 c) 35

d) 20 e) 15 f) none of the above



Use 4), 5), 6) to fill in the chart this far.

$n(A \cap C)$ is not supplied. So use x to represent "?"



The chart now looks like this, using 1), 2), + 3)

By 7) $n(A \cup B \cup C) = 190$

By the chart: $n(A \cup B \cup C) = \underbrace{\underbrace{100}_{n(A)} + 50 + 10 + 55 - x}$

$\Rightarrow \quad 190 = 215 - x$

$\Rightarrow \quad 25 = x \qquad \Rightarrow n(A \cap C) = 25 + 5 = 30$

15) From a group of 10 flowers you pick 7 to take home. In how many ways can this be done? Order of choice doesn't matter.

Examples: One way is to select flowers 1,2,4,5,8,9 and 10. Another is to select is to select flowers 2,3,4,5,7,8,9.

a) 720 b) 360 c) 120

d) 60 e) 256 f) none of the above

From 10 objects choose 7 without regard to order. There are C(10,7) ways to do this.

$$C(10,7) = \frac{10!}{(10-7)!\,7!} = \frac{10 \cdot 9 \cdot 8}{(10-7)!} = \frac{10 \cdot 9 \cdot 8}{3!}$$

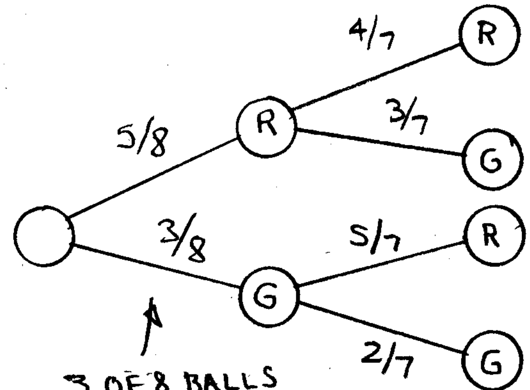
$$= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 10 \cdot 3 \cdot 4 = \underline{\underline{120}}$$

16) An urn contains 5 red balls and 3 green. Two balls are selected from the urn, one after the other without replacement. Given that two balls of the same color were drawn out, what is the probability that they are both red?

a) 10/13 b) 5/8 c) 1/2

d) 5/14 e) 13/28 f) none of the above

Draw a tree



$\frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56}$ 2R 2C

$\frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56}$. SC

3 OF 8 BALLS ARE GREEN

AFTER DRAWING OUT 1 GREEN, 2 OF 7 BALLS ARE GREEN

SAME COLOR

The problem asks for $\Pr(2R \mid SC) = \dfrac{\Pr(2R \cap SC)}{\Pr(SC)}$

Now $2R \cap SC = 2R$. $\Rightarrow$ $= \dfrac{\Pr(2R)}{\Pr(SC)}$

Using the tree $= \dfrac{20/56}{20/56 + 6/56}$

$= 20/26 = \underline{\underline{10/13}}$

17) A multiple choice test consists of 4 questions, each question with 5 choices for an answer, and each question has exactly 1 correct answer. A student takes the test and marks each question completely at random (with each choice for an answer being made independently of the other choices for answers). What is the probability that the student gets exactly one problem correct?

a) 12/625 b) 20/256 c) 256/625

d) 4/3125 e) 4/625 f) none of the above

This is a Bernoulli process.

$$Pr(1 \text{ correct}) = \underset{\substack{\uparrow \\ \text{COULD GET} \\ 1^{st} \text{ correct or } 2^{nd} \text{ or } 3^{rd} \text{ or } 4^{th}}}{4} \underbrace{\tfrac{4}{5} \cdot \tfrac{4}{5} \cdot \tfrac{4}{5} \cdot \tfrac{1}{5}}_{Pr(\text{getting only the last question correct})}$$

$$= \frac{4^4}{5^4} = \frac{256}{625}$$

EQUIVALENTLY

4 questions

$$Pr(1 \text{ correct}) = \underset{\text{one correct}}{\binom{4}{1}} \underset{\substack{Pr(\text{INCORRECT}) \\ \text{ON ONE PROBLEM}}}{\left(\tfrac{4}{5}\right)^3} \left(\tfrac{1}{5}\right)^1$$

$\left(\tfrac{1}{5}\right)$ ← Pr(CORRECT)

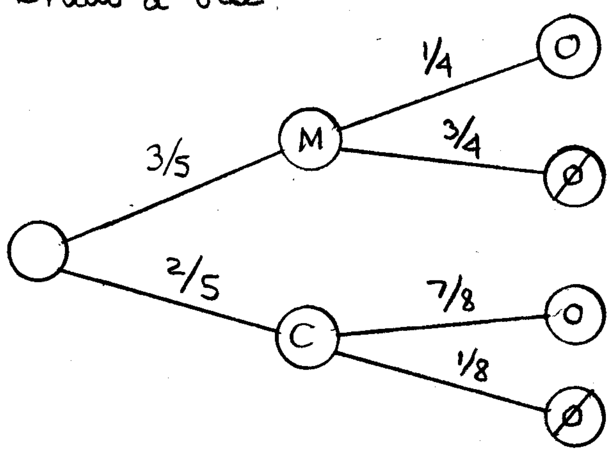
18) A fish hatchery tags one of the new, genetically engineered silver streak salmon in its stock. Then the hatchery delivers 3/5 of its entire stock of silver streak salmon to Moon Lake. The remaining 2/5 are delivered to Clear Lake*. Of those silver streak salmon delivered to Moon Lake, 1/4 will eventually make it to the open ocean. Of those silver streak salmon delivered to Clear Lake, 7/8 will eventually make it to the open ocean. Given that the tagged salmon made it to the ocean, what is the probability that it was originally delivered to Moon Lake?

*With probability 3/5 the tagged salmon is delivered to Moon Lake (and with probability 2/5 to Clear Lake).

a) 3/4　　b) 1/4　　c) 9/16

d) 3/10　　e) 3/20　　f) none of the above

Draw a tree:



	M	O
$\frac{3}{5} \cdot \frac{1}{4} = \frac{3}{20}$	✓	✓
$\frac{3}{5} \cdot \frac{3}{4} = \frac{9}{20}$	✓	
$\frac{2}{5} \cdot \frac{7}{8} = \frac{7}{20}$		✓
$\frac{2}{5} \cdot \frac{1}{8} = \frac{1}{20}$		

$= 1$ CHECK ADDS UP TO ONE

$$Pr(M|O) = \frac{Pr(M \cap O)}{Pr(O)}$$

$$= \frac{3/20}{3/20 + 7/20} = \underline{\underline{3/10}}$$

19) A vase contains 3 red flowers and 6 blue flowers and 1 yellow flower. Completely at random, a flower is drawn out. Let X = #blue flowers - #red flowers remaining in the vase (i.e. number of blue flowers minus the number of red flowers <u>still</u> in the vase <u>after</u> one flower is drawn out). Find $E(X)$.

a) 3.1　　b) 2.7　　c) 2

d) 3　　e) 2.6　　f) none of the above

Case I: One red flower is drawn out. Then 2R, 6B, 1Y are in vase afterwards. So $X = 6-2=4$,

$X=4$ $\qquad \Pr(\text{Case I}) = 3/10$ (3 of 10 flowers are R).

Case II: One blue flower is drawn out, leaving 3R, 5B, 1Y.

$\Rightarrow X = 5-3=2, \qquad \Pr(\text{Case II}) = 6/10$

Case III: One yellow flower is drawn out leaving 3R, 6B.

$\Rightarrow X = 6-3=3. \qquad \Pr(\text{Case III}) = 1/10$

So $\Pr(X=4) = 3/10$, $\Pr(X=2) = 6/10$, $\Pr(X=3) = 1/10$

$$E(X) = 4 \cdot 3/10 + 2 \cdot 6/10 + 3 \cdot 1/10$$

$$= \frac{12+12+3}{10} = \frac{27}{10} = \underline{\underline{2.7}}$$

20) How many 7 letter "words" can be formed using the 7 letters EEEEDDX (i.e. 4 E's and 2 D's and 1 X)? Example: EDEXDEE.

a) 48 b) 210 c) 35

d) 96 e) 105 f) none of the above

$$\frac{7!}{4!\,2!}$$

# of ways if the letters were distinct

ways to permute the D's.

4! ways to permute the E's leaving the word unchanged

$$= \frac{7\cdot 6\cdot 5}{2} = \underline{\underline{105}}$$

21) An incomplete deck of cards consists of 4 aces, 4 twos, 4 threes, 4 fours, and 4 fives (i.e. this is a complete deck stripped of all face cards and all 6's, 7's, 8's, 9's, 10's). You are dealt 3 cards from this incomplete deck. How many three card hands are possible where all three cards are of different values? Aces have a value of 1. Order doesn't matter. Example: If you are dealt a 5♡ and a 3♣ and an A♡ (ace of hearts), these three cards are of different values.

a) 640 b) 1180 c) 320

d) 540 e) 270 f) none of the above

I): Select the 3 values (from 5 possible). There are $C(5,3)$ ways to do this. $C(5,3) = 10$.

II). From I), value 1, value 2, value 3 have been selected (e.g. A's, 4's, 5's)

From the 4 cards of value 1 select 1 card. There are 4 ways to do this.
From the 4 cards of value 2 select 1 card. There are 4 ways to do this.
From the 4 cards of value 3 select 1 card. There are 4 ways to do this.

Answer: $\binom{5}{3} \cdot 4 \cdot 4 \cdot 4 = 10 \cdot 64 = \underline{\underline{640}}$

22) A parking lot contains 4 Ferraris, 2 Maseratis, and 5 Porsches. You get to select two of the cars to take home. How many choices* do you have provided that you do not choose 2 Porsches? Order of choice doesn't matter.

*Example: One possible choice is F2 and P4 (Ferrari #2 and Porsche #4). Another possible choice is M2 and P1.

a) 17	b) 38	c) 48
d) 45	e) 55	f) none of the above

With no restrictions there are $C(11,2)$ ways to choose 2 cars.

From 11 cars choose 2.

There are $C(5,2)$ ways to choose 2 Porsches. These

From 5 Porsches select 2.

ways are disallowed in the statement of the problem.

Answer: $C(11,2) - C(5,2) = 55 - 10 = \underline{\underline{45}}$

23) A kid notices that when her piggy bank is shaken, a penny comes out twice as often as a nickel, four times as often as a dime, and four times as often as a quarter. There are no other coins in the bank besides pennies, nickels, dimes, and quarters. What is the probability that a quarter will fall out if the kid shakes her piggy bank?

a) 1/2 b) 1/8 c) 1/4

d) 1/16 e) 3/16 f) none of the above

Using obvious notation

$$Pr(P) = 2\,Pr(N) = 4\,Pr(D) = 4\,Pr(Q)$$

The probability of getting a penny is twice that of getting a nickel

So
$$\left.\begin{aligned} Pr(N) &= \tfrac{1}{2}\,Pr(P) \\ Pr(D) &= \tfrac{1}{4}\,Pr(P) \\ Pr(Q) &= \tfrac{1}{4}\,Pr(P) \end{aligned}\right\} \star$$

However: $* \quad Pr(P) + Pr(N) + Pr(D) + Pr(Q) = 1$

You will get one of P, N, D, Q.

Substituting $\star$ into $*$:

$$Pr(P) + \tfrac{1}{2}Pr(P) + \tfrac{1}{4}Pr(P) + \tfrac{1}{4}Pr(P) = 1$$

$$\Rightarrow \quad 2\,Pr(P) = 1 \quad \Rightarrow \quad Pr(P) = \tfrac{1}{2}$$

$$\Rightarrow \quad Pr(Q) = \tfrac{1}{4}\,Pr(P) = \tfrac{1}{4}\cdot\tfrac{1}{2} = \underline{\underline{\tfrac{1}{8}}}$$

24) Suppose $Pr[A] = .7$ and $Pr[B] = .2$, and that A and B are <u>independent</u> events. If possible, find $Pr[A \cup B]$.

a) .56 b) .9 c) .76

d) .06 e) Can't be determined from the given information. f) none of the above

If A and B are independent, then

$$Pr(A \cap B) = Pr(A) \cdot Pr(B)$$

$$\Rightarrow Pr(A \cap B) = .7 \cdot .2 = .14$$

So

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

$$= .7 + .2 - .14$$

$$= \underline{\underline{.76}}$$

25) Six tomatoes sit on a window ledge. One tomato has no worms in it. Two tomatoes each have one worm in them. And three tomatoes each have two worms. Two tomatoes are selected at random. Let X = the **total** number of worms in the two chosen tomatoes. Find $Pr[X = 2]$.

a) 1/3 b) 1/15 c) 1/6

d) 4/15 e) 2/15 f) none of the above

$(X=2)$ is the event of getting 2 worms. This can occur by:

I) Getting one tomato with 0 worms and one with 2 worms.

$$Pr(0 \text{ worms } \& \ 2 \text{ worms}) = \frac{\binom{1}{1}\binom{3}{1}}{\binom{6}{2}}$$

ways to choose a 0 worm tomato

ways to choose 1 of the 3 tomatoes that have 2 worms.

$\boxed{\text{*NOTATION FOR } C(6,2)} \longrightarrow \binom{6}{2}$

ways to choose 2 of 6 tomatoes

II) Getting 2 tomatoes with one worm in them each

$$Pr(1 \text{ worm } \& \ 1 \text{ worm}) = \frac{\binom{2}{2}}{\binom{6}{2}}$$

$\binom{2}{2}$ ← Choose both tomatoes with 1 worm in them. There is only one way to do this

So

$$Pr[X=2] = \frac{\binom{1}{1}\binom{3}{1}}{\binom{6}{2}} + \frac{\binom{2}{2}}{\binom{6}{2}} = \frac{3+1}{\binom{6}{2}} = \frac{4}{6\cdot 5/2}$$

$$= \underline{\underline{4/15}}$$

*<u>NOTATION:</u> $\binom{n}{r} = C(n,r)$