Homework Problem Set 12

(10.1.3) 1. Let p be a probability distribution on $\{0, 1, 2\}$ with moments $\mu_1 = 1$, $\mu_2 = 3/2$.

(a) Find its ordinary generating function $h(z)$.

(b) Using (a), find its moment generating function.

(c) Using (b), find its first six moments.

(d) Using (a), find p_0, p_1, and p_2.

(10.1.7) 2. Let X be a discrete random variable with values in $\{0, 1, 2, \ldots, n\}$ and moment generating function $g(t)$. Find, in terms of $g(t)$, the generating functions for

(a) $-X$.

(b) $X + 1$.

(c) $3X$.

(d) $aX + b$.

(10.1.11) 3. Show that if X is a random variable with mean μ and variance σ^2, and if $X^* = (X - \mu)/\sigma$ is the standardized version of X, then

$$g_{X^*}(t) = e^{-\mu t/\sigma} g_X\left(\frac{t}{\sigma}\right) .$$

(10.2.2) 4. Let Z_1, Z_2, $\ldots$, Z_N describe a branching process in which each parent has j offspring with probability p_j. Find the probability d that the process dies out if

(a) $p_0 = 1/2$, $p_1 = p_2 = 0$, and $p_3 = 1/2$.

(b) $p_0 = p_1 = p_2 = p_3 = 1/4$.

(c) $p_0 = t$, $p_1 = 1 - 2t$, $p_2 = 0$, and $p_3 = t$, where $t \leq 1/2$.

(10.2.4) 5. Let $S_N = X_1 + X_2 + \cdots + X_N$, where the X_i's are independent random variables with common distribution having generating function $f(z)$. Assume that N is an integer valued random variable independent of all of the X_j and having generating function $g(z)$. Show that the generating function for S_N is $h(z) = g(f(z))$. *Hint*: Use the fact that

$$h(z) = E(z^{S_N}) = \sum_k E(z^{S_N}|N = k)P(N = k) .$$

(Rice 5.6) 6. Using moment-generating functions, show that as $\alpha \to \infty$ the gamma distribution with parameters α and λ, properly standardized, tends to the standard normal distribution.

(10.3.8) 7. Let X_1, X_2, $\ldots$, X_n be an independent trials process with uniform density. Find the moment generating function for

(a) X_1.

(b) $S_2 = X_1 + X_2$.

(c) $S_n = X_1 + X_2 + \cdots + X_n$.

(d) $A_n = S_n/n$.

(e) $S_n^* = (S_n - n\mu)/\sqrt{n\sigma^2}$.

(Rice 5.28) 8. Let f_n be a sequence of frequency functions with $f_n(x) = 1/2$ if $x = \pm(1/2)^n$ and $f_n(x) = 0$ otherwise. Show that $\lim f_n(x) = 0$ for all x, which means that the frequency functions do not converge to a frequency function, but that there exists a cumulative distribution function F such that $\lim F_n(x) = F(x)$.

(Rice 5.29) 9. In addition to limit theorems that deal with sums, there are limit theorems that deal with extreme values such as maxima or minima. Here is an example. Let $U_1, ..., U_n$ be independent uniform random variables on $[0, 1]$, and let $U_{(n)}$ be the maximum. Find the cumulative distribution function of $U_{(n)}$ and a standardized $U_{(n)}$, and show that the cumulative distribution function of the standardized variable tends to a limiting value.

10. Assume that X is uniform on $[0, 1]$. Let $Y = \sqrt{X}$. Find the approximate mean and variance of Y and find the exact mean and variance of Y and compare the two sets of numbers.