

Homework Problem Set 10

- (7.1.10) 1. (Lévy¹) Assume that n is an integer, not prime. Show that you can find two distributions a and b on the nonnegative integers such that the convolution of a and b is the equiprobable distribution on the set $0, 1, 2, \dots, n-1$. If n is prime this is not possible, but the proof is not so easy. (Assume that neither a nor b is concentrated at 0.)
2. Use convolutions to find the distribution of the sum of two independent random variables that are both uniform on the integers $\{0, 1, 2, \dots, n-1\}$.
3. Use convolutions to prove that the sum of two independent Poisson random variables, one with rate λ and one with rate μ , is itself Poisson with rate $\lambda + \mu$.
4. Use convolutions to prove that the sum of two independent Binomial random variables, each with success probability p , but one with n trials and the other with m trials, is $\text{Bin}(n + m, p)$.
5. Use convolutions to prove that the sum of two independent negative binomial random variables, each with the same success probability p , also has a negative binomial distribution.
6. Use convolutions to prove that the sum of two independent Gamma distributed random variables, each with the same rate λ , is also Gamma distributed with the sum of the two shapes as the shape of the sum.
7. Suppose that $R^2 = X^2 + Y^2$. Find f_{R^2} and f_R if X and Y are independent standard normal distributions.
- (7.2.10) 8. Let X_1, X_2, \dots, X_n be n independent random variables each of which has an exponential density with mean μ . Let M be the *minimum* value of the X_j . Show that the density for M is exponential with mean μ/n . *Hint*: Use cumulative distribution functions.
- (7.2.13) 9. Particles are subject to collisions that cause them to split into two parts with each part a fraction of the parent. Suppose that this fraction is uniformly distributed between 0 and 1. Following a single particle through several splittings we obtain a fraction of the original particle $Z_n = X_1 \cdot X_2 \cdot \dots \cdot X_n$ where each X_j is uniformly distributed between 0 and 1. Show that the density for the random variable Z_n is

$$f_n(z) = \frac{1}{(n-1)!} (-\log z)^{n-1}.$$

Hint: Show that $Y_k = -\log X_k$ is exponentially distributed. Use this to find the density function for $S_n = Y_1 + Y_2 + \dots + Y_n$, and from this the cumulative distribution and density of $Z_n = e^{-S_n}$.

- (7.2.14) 10. Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ . Show that $Z = X_1 - X_2$ has density

$$f_Z(z) = (1/2)\lambda e^{-\lambda|z|}.$$

¹See M. Krasner and B. Ranulac, "Sur une Propriété des Polynomes de la Division du Cercle"; and the following note by J. Hadamard, in *C. R. Acad. Sci.*, vol. 204 (1937), pp. 397-399.