Homework Problem Set 7

(5.2.7) 1. Explain how you can generate a random variable whose cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x^2, & \text{if } 0 \le x \le 1, \\ 1, & \text{if } x > 1. \end{cases}$$

(5.2.9) 2. Let U, V be random numbers chosen independently from the interval $[0,1]$ with uniform distribution. Find the cumulative distribution and density of each of the variables

(a) $Y = U + V$.

(b) $Y = |U - V|$.

(5.2.10) 3. Let U, V be random numbers chosen independently from the interval $[0,1]$. Find the cumulative distribution and density for the random variables

(a) $Y = \max(U, V)$.

(b) $Y = \min(U, V)$.

(5.2.12) 4. A number U is chosen at random in the interval $[0,1]$. Find the probability that

(a) $R = U^2 < 1/4$.

(b) $S = U(1 - U) < 1/4$.

(c) $T = U/(1 - U) < 1/4$.

(5.2.16) 5. Let X be a random variable with density function

$$f_X(x) = \begin{cases} cx(1-x), & \text{if } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function F_X for X?

(c) What is the probability that $X < 1/4$?

(5.2.21) 6. Let X be a random variable with cumulative distribution function F strictly increasing on the range of X. Let $Y = F(X)$. Show that Y is uniformly distributed in the interval $[0,1]$. (The formula $X = F^{-1}(Y)$ then tells us how to construct X from a uniform random variable Y.)

Test Score	Letter grade
$\mu + \sigma < x$	A
$\mu < x < \mu + \sigma$	B
$\mu - \sigma < x < \mu$	C
$\mu - 2\sigma < x < \mu - \sigma$	D
$x < \mu - 2\sigma$	F

Table 1: Grading on the curve.

(5.2.27) 7. A final examination at Podunk University is constructed so that the test scores are approximately normally distributed, with parameters μ and σ. The instructor assigns letter grades to the test scores as shown in Table 1 (this is the process of "grading on the curve").

(5.2.34) 8. Jones puts in two new lightbulbs: a 60 watt bulb and a 100 watt bulb. It is claimed that the lifetime of the 60 watt bulb has an exponential density with average lifetime 200 hours ($\lambda = 1/200$). The 100 watt bulb also has an exponential density but with average lifetime of only 100 hours ($\lambda = 1/100$). Jones wonders what is the probability that the 100 watt bulb will outlast the 60 watt bulb.

If X and Y are two independent random variables with exponential densities $f(x) = \lambda e^{-\lambda x}$ and $g(x) = \mu e^{-\mu x}$, respectively, then the probability that X is less than Y is given by

$$P(X < Y) = \int_0^\infty f(x)(1 - G(x))\, dx,$$

where $G(x)$ is the cumulative distribution function for $g(x)$. Explain why this is the case. Use this to show that

$$P(X < Y) = \frac{\lambda}{\lambda + \mu}$$

and to answer Jones's question.

(5.2.36) 9. Let X be a random variable having an exponential density with parameter λ. Find the density for the random variable $Y = rX$, where r is a positive real number.

(5.2.37) 10. Let X be a random variable having a normal density and consider the random variable $Y = e^X$. Then Y has a *log normal* density. Find this density of Y.