Homework Problem Set 5

(4.2.2) 1. A radioactive material emits α-particles at a rate described by the density function

$$f(t) = .1e^{-.1t} .$$

Find the probability that a particle is emitted in the first 10 seconds, given that

(a) no particle is emitted in the first second.

(b) no particle is emitted in the first 5 seconds.

(c) a particle is emitted in the first 3 seconds.

(d) a particle is emitted in the first 20 seconds.

(4.2.5) 2. Suppose you choose two numbers x and y, independently at random from the interval $[0, 1]$. Given that their sum lies in the interval $[0, 1]$, find the probability that

(a) $|x - y| < 1$.

(b) $xy < 1/2$.

(c) $\max\{x, y\} < 1/2$.

(d) $x^2 + y^2 < 1/4$.

(e) $x > y$.

(4.2.6) 3. Find the conditional density functions for the following experiments.

(a) A number x is chosen at random in the interval $[0, 1]$, given that $x > 1/4$.

(b) A number t is chosen at random in the interval $[0, \infty)$ with exponential density e^{-t}, given that $1 < t < 10$.

(c) A dart is thrown at a circular target of radius 10 inches, given that it falls in the upper half of the target.

(d) Two numbers x and y are chosen at random in the interval $[0, 1]$, given that $x > y$.

(Rice 3.24) 4. Let P have a uniform distribution on$[0, 1]$, and, conditional on $P = p$, let X have a Bernoulli distribution with parameter p. (That is, $X = 1$ with probability p and $X = 0$ with probability $1 - p$.) Find the conditional distribution of P given X.

(4.2.8) 5. Let x and y be chosen at random from the interval $[0, 1]$. Which pairs of the following events are independent?

(a) $x > 1/3$.

(b) $y > 2/3$.

(c) $x > y$.

(d) $x + y < 1$.

(4.2.11) 6. A coin has an unknown bias p that is assumed to be uniformly distributed between 0 and 1. The coin is tossed n times and heads turns up j times and tails turns up k times. We have seen that the probability that heads turns up next time is

$$\frac{j+1}{n+2} .$$

Show that this is the same as the probability that the next ball is black for the Polya urn model of Exercise 4.1.20. Use this result to explain why, in the Polya urn model, the proportion of black balls does not tend to 0 or 1 as one might expect but rather to a uniform distribution on the interval $[0, 1]$.

(4.3.1) 7. One of the first conditional probability paradoxes was provided by Bertrand.[1] It is called the *Box Paradox*. A cabinet has three drawers. In the first drawer there are two gold balls, in the second drawer there are two silver balls, and in the third drawer there is one silver and one gold ball. A drawer is picked at random and a ball chosen at random from the two balls in the drawer. Given that a gold ball was drawn, what is the probability that the drawer with the two gold balls was chosen?

(4.3.2) 8. The following problem is called the *two aces problem*. This problem, dating back to 1936, has been attributed to the English mathematician J. H. C. Whitehead (see Gridgeman[2]). This problem was also submitted to Marilyn vos Savant by the master of mathematical puzzles Martin Gardner, who remarks that it is one of his favorites.

A bridge hand has been dealt, i. e. thirteen cards are dealt to each player. Given that your partner has at least one ace, what is the probability that he has at least two aces? Given that your partner has the ace of hearts, what is the probability that he has at least two aces? Answer these questions for a version of bridge in which there are eight cards, namely four aces and four kings, and each player is dealt two cards. (The reader may wish to solve the problem with a 52-card deck.)

(4.3.3) 9. Solve the previous problem with a 52-card deck.

(4.3.3) 10. In the preceding exercise(s), it is natural to ask "How do we get the information that the given hand has an ace?" Gridgeman considers two different ways that we might get this information. (Again, assume the deck consists of eight cards.)

(a) Assume that the person holding the hand is asked to "Name an ace in your hand" and answers "The ace of hearts." What is the probability that he has a second ace?

(b) Suppose the person holding the hand is asked the more direct question "Do you have the ace of hearts?" and the answer is yes. What is the probability that he has a second ace?

[1] J. Bertrand, *Calcul des Probabilités*, Gauthier-Uillars, 1888.

[2] N. T. Gridgeman, Letter, *American Statistician*, 21 (1967), pgs. 38-39.