Homework Problem Set 4

1. Use a counting argument to prove that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

2. Use a counting argument to prove that

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

3. Use a counting argument to prove that

$$n^3 = 1 + 3(n-1) + 3(n-1)^2 + (n-1)^3$$

4. Use a counting argument to prove that

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

(4.1.7) 5. A coin is tossed twice. Consider the following events.
A: Heads on the first toss.
B: Heads on the second toss.
C: The two tosses come out the same.

(a) Show that A, B, C are pairwise independent but not independent.

(b) Show that C is independent of A and B but not of $A \cap B$.

(4.1.18) 6. A doctor assumes that a patient has one of three diseases d_1, d_2, or d_3. Before any test, he assumes an equal probability for each disease. He carries out a test that will be positive with probability .8 if the patient has d_1, .6 if he has disease d_2, and .4 if he has disease d_3. Given that the outcome of the test was positive, what probabilities should the doctor now assign to the three possible diseases?

(4.1.27) 7. (Chung[1]) In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted that it won't rain, are both equal to 2/3. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability 1/3. Find

(a) the probability that Pickwick has no umbrella, given that it rains.

(b) the probability that he brings his umbrella, given that it doesn't rain.

(4.1.28) 8. Probability theory was used in a famous court case: *People v. Collins.*[2] In this case a purse was snatched from an elderly person in a Los Angeles suburb. A couple seen running from the scene were described as a black man with a beard and a mustache and a blond girl with hair in a ponytail. Witnesses said they drove off in a partly

[1] K. L. Chung, *Elementary Probability Theory With Stochastic Processes, 3rd ed.* (New York: Springer-Verlag, 1979), p. 152.

[2] M. W. Gray, "Statistics and the Law," *Mathematics Magazine,* vol. 56 (1983), pp. 67–81.

yellow car. Malcolm and Janet Collins were arrested. He was black and though clean shaven when arrested had evidence of recently having had a beard and a mustache. She was blond and usually wore her hair in a ponytail. They drove a partly yellow Lincoln. The prosecution called a professor of mathematics as a witness who suggested that a conservative set of probabilities for the characteristics noted by the witnesses would be as shown in Table 1.

man with mustache	1/4
girl with blond hair	1/3
girl with ponytail	1/10
black man with beard	1/10
interracial couple in a car	1/1000
partly yellow car	1/10

Table 1: Collins case probabilities.

The prosecution then argued that the probability that all of these characteristics are met by a randomly chosen couple is the product of the probabilities or 1/12,000,000, which is very small. He claimed this was proof beyond a reasonable doubt that the defendants were guilty. The jury agreed and handed down a verdict of guilty of second-degree robbery.

If you were the lawyer for the Collins couple how would you have countered the above argument? (The appeal of this case is discussed in Exercise 5.1.34.)

(4.1.39) 9. Assume that the random variables X and Y have the joint distribution given in Table 2.

		Y			
		-1	0	1	2
X	-1	0	1/36	1/6	1/12
	0	1/18	0	1/18	0
	1	0	1/36	1/6	1/12
	2	1/12	0	1/12	1/6

Table 2: Joint distribution.

(a) What is $P(X \geq 1 \text{ and } Y \leq 0)$?

(b) What is the conditional probability that $Y \leq 0$ given that $X = 2$?

(c) Are X and Y independent?

(d) What is the distribution of $Z = XY$?

(4.1.62) 10. (a) Suppose that you are looking in your desk for a letter from some time ago. Your desk has eight drawers, and you assess the probability that it is in any particular drawer is 10% (so there is a 20% chance that it is not in the desk at all). Suppose now that you start searching systematically through your desk, one drawer at a time. In addition, suppose that you have not found the letter in the first i drawers, where $0 \leq i \leq 7$. Let p_i denote the probability that the letter will be found in the next drawer, and let q_i denote the probability that the letter will be found in some subsequent drawer (both p_i and q_i are conditional probabilities, since they are based upon the assumption that the letter is not in the first i drawers). Show that the p_i's increase and the q_i's decrease. (This problem is from Falk et al.[3])

(b) The following data appeared in an article in the Wall Street Journal.[4] For the ages 20, 30, 40, 50, and 60, the probability of a woman in the U.S. developing cancer in the next ten years is 0.5%, 1.2%, 3.2%, 6.4%, and 10.8%, respectively. At the same set of ages, the probability of a woman in the U.S. eventually developing cancer is 39.6%, 39.5%, 39.1%, 37.5%, and 34.2%, respectively. Do you think that the problem in part (a) gives an explanation for these data?

[3] R. Falk, A. Lipson, and C. Konold, "The ups and downs of the hope function in a fruitless search," in *Subjective Probability,* G. Wright and P. Ayton, (eds.) (Chichester: Wiley, 1994), pgs. 353-377.

[4] C. Crossen, "Fright by the numbers: Alarming disease data are frequently flawed," *Wall Street Journal,* 11 April 1996, p. B1.