Homework Problem Set 1

(1.1.5) 1. Consider the bet that all three dice will turn up sixes at least once in n rolls of three dice. Calculate $f(n)$, the probability of at least one triple-six when three dice are rolled n times. Determine the smallest value of n necessary for a favorable bet that a triple-six will occur when three dice are rolled n times. (DeMoivre would say it should be about $216 \log 2 = 149.7$ and so would answer 150—see Exercise 1.2.16. Do you agree with him?)

(1.2.7) 2. Let A and B be events such that $P(A \cap B) = 1/4$, $P(\tilde{A}) = 1/3$, and $P(B) = 1/2$. What is $P(A \cup B)$?

(1.2.13) 3. In a horse race, the odds that Romance will win are listed as $2:3$ and that Downhill will win are $1:2$. What odds should be given for the event that either Romance or Downhill wins?

(1.2.14) 4. Let X be a random variable with distribution function $m_X(x)$ defined by

$$m_X(-1) = 1/5, \quad m_X(0) = 1/5, \quad m_X(1) = 2/5, \quad m_X(2) = 1/5 \; .$$

(a) Let Y be the random variable defined by the equation $Y = X + 3$. Find the distribution function $m_Y(y)$ of Y.

(b) Let Z be the random variable defined by the equation $Z = X^2$. Find the distribution function $m_Z(z)$ of Z.

(1.2.17) 5. Assume that the probability of a "success" on a single experiment with n outcomes is $1/n$. Let m be the number of experiments necessary to make it a favorable bet that at least one success will occur (see Exercise 1.1.5).

(a) Show that the probability that, in m trials, there are no successes is $(1 - 1/n)^m$.

(b) (de Moivre) Show that if $m = n \log 2$ then

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^m = \frac{1}{2} \; .$$

Hint:

$$\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \; .$$

Hence for large n we should choose m to be about $n \log 2$.

(c) Would DeMoivre have been led to the correct answer for de Méré's two bets if he had used his approximation?

(1.2.18) 6. (a) For events $A_1, \ldots, A_n$, prove that

$$P(A_1 \cup \cdots \cup A_n) \le P(A_1) + \cdots + P(A_n) \; .$$

(b) For events A and B, prove that

$$P(A \cap B) \ge P(A) + P(B) - 1.$$

(1.2.23) 7. Let Ω be the sample space

$$\Omega = \{0, 1, 2, \dots\} ,$$

and define a distribution function by

$$m(j) = (1 - r)^j r ,$$

for some fixed r, $0 < r < 1$, and for $j = 0, 1, 2, \dots$. Show that this is a distribution function for Ω.

(1.2.25) 8. Tversky and Kahneman[1] asked a group of subjects to carry out the following task. They are told that:

> Linda is 31, single, outspoken, and very bright. She majored in philosophy in college. As a student, she was deeply concerned with racial discrimination and other social issues, and participated in anti-nuclear demonstrations.

The subjects are then asked to rank the likelihood of various alternatives, such as:
(1) Linda is active in the feminist movement.
(2) Linda is a bank teller.
(3) Linda is a bank teller and active in the feminist movement.

Tversky and Kahneman found that between 85 and 90 percent of the subjects rated alternative (1) most likely, but alternative (3) more likely than alternative (2). Is it? They call this phenomenon the *conjunction fallacy,* and note that it appears to be unaffected by prior training in probability or statistics. Is this phenomenon a fallacy? If so, why? Can you give

(1.2.28) 9. Here is an attempt to get around the fact that we cannot choose a "random integer."

(a) What, intuitively, is the probability that a "randomly chosen" positive integer is a multiple of 3?

(b) Let $P_3(N)$ be the probability that an integer, chosen at random between 1 and N, is a multiple of 3 (since the sample space is finite, this is a legitimate probability). Show that the limit

$$P_3 = \lim_{N \to \infty} P_3(N)$$

exists and equals 1/3. This formalizes the intuition in (a), and gives us a way to assign "probabilities" to certain events that are infinite subsets of the positive integers.

(c) If A is any set of positive integers, let $A(N)$ mean the number of elements of A which are less than or equal to N. Then define the "probability" of A as

$$P(A) = \lim_{N \to \infty} A(N)/N ,$$

provided this limit exists. Show that this definition would assign probability 0 to any finite set and probability 1 to the set of all positive integers. Thus, the probability of the set of all integers is not the sum of the probabilities of the individual integers in this set. This means that the definition of probability given here is not a completely satisfactory definition.

[1] K. McKean, "Decisions, Decisions," pp. 22–31.

(d) Let A be the set of all positive integers with an odd number of digits. Show that $P(A)$ does not exist. This shows that under the above definition of probability, not all sets have probabilities.

(1.2.31) 10. (from vos Savant[2]) A reader of Marilyn vos Savant's column wrote in with the following question:

> My dad heard this story on the radio. At Duke University, two students had received A's in chemistry all semester. But on the night before the final exam, they were partying in another state and didn't get back to Duke until it was over. Their excuse to the professor was that they had a flat tire, and they asked if they could take a make-up test. The professor agreed, wrote out a test and sent the two to separate rooms to take it. The first question (on one side of the paper) was worth 5 points, and they answered it easily. Then they flipped the paper over and found the second question, worth 95 points: 'Which tire was it?' What was the probability that both students would say the same thing? My dad and I think it's 1 in 16. Is that right?"

(a) Is the answer 1/16?

(b) The following question was asked of a class of students. "I was driving to school today, and one of my tires went flat. Which tire do you think it was?" The responses were as follows: right front, 58%, left front, 11%, right rear, 18%, left rear, 13%. Suppose that this distribution holds in the general population, and assume that the two test-takers are randomly chosen from the general population. What is the probability that they will give the same answer to the second question?

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[2]M. vos Savant, *Parade Magazine*, 3 March 1996, p. 14.