Practice Exam 2

1: Give the definitions of the following terms:

(a) Recurrent state

(b) Transient state

(c) Accessible

(d) Communicates

(e) Equivalence relation

(f) Periodic state

(g) Ergodic

(h) Regular

(i) Absorbing

2: Prove that communicating is an equivalence relation.

3: Which of the following Markov chains are regular? Explain your answers.

$$(A)\begin{bmatrix} .1 & .2 & .3 & .3 & .1 \\ .2 & .2 & .2 & .2 & .2 \\ .5 & .1 & .1 & .2 & .1 \\ .6 & .1 & .1 & .1 & .1 \\ .1 & .1 & .2 & .2 & .4 \end{bmatrix} \qquad (B)\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & .4 & .3 \\ 0 & 0 & .4 & .3 & .3 \\ 0 & 0 & .3 & .3 & .4 \end{bmatrix} \qquad (C)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ .2 & .2 & .2 & .2 & .2 \\ .2 & .2 & .2 & .2 & .2 \\ .2 & .2 & .2 & .2 & .2 \\ .2 & .2 & .2 & .2 & .2 \end{bmatrix}$$

$$(D)\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ .5 & 0 & .5 & 0 & 0 \end{bmatrix} \qquad (E)\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ .3 & 0 & .3 & 0 & .4 \end{bmatrix}$$

4: Classify each of the seven states in the following Markov chain:

$$\begin{bmatrix} .1 & .1 & .2 & .1 & .2 & .1 & .2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5: Find the stationary distribution for the following regular Markov chain:

$$\begin{bmatrix} .2 & .2 & .6 \\ .1 & .9 & 0 \\ 0 & .5 & .5 \end{bmatrix}$$

6: Answer the following questions for the Markov chain given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ .1 & .2 & 0 & .2 & .5 \\ 0 & 0 & .1 & .6 & .3 \end{bmatrix}$$

A: Given that you start in state 4, what is the expected number of visits to state 5 before you are absorbed into one of the states 1, 2, or 3?

B: Given that you start in state 5, what is the probability you are absorbed into state 1?

7: A mutant version of a gene arises in a tiny population of 4 individuals. In this case, think of the states as 0, 1, 2, and 3 where the number corresponds to the number of individuals with the mutant gene. We start in state 1 which corresponds to the second row of the transition matrix given below. State 0 corresponds to the loss of the mutant gene from the population. State 3 corresponds to the fixation of the mutant gene in the population.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .4 & 0 & .6 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the probability that the gene will become fixed in the population?

8: Consider two urns, urn 1 and urn 2. Initially, urn 1 contains 2 green balls and 1 red ball while urn 2 contains 2 red balls and 1 green ball. A ball is chosen at random from each urn and then the balls are exchanged between the urns. What is the probability urn 1 will contain all red balls before it contains all green balls?

9: An unfair coin with $\Pr[H] = 0.6$ is flipped repeatedly until there are either two consecutive heads or two consecutive tails. Formulate this problem as a Markov chain by considering the states to be the outcomes of adjacent coin tosses. Which states are absorbing? Give the transition matrix P for this chain. Use the fundamental matrix along with your initial probabilities of being in each state to determine explicitly what the expected number of flips will be. This problem is a little challenging.

10: Suppose fish come in 3 sizes: small, medium, and large, and that the size-stratified population model is given by the following matrix of vital rates:

$$\begin{bmatrix} 0 & 2 & 6 \\ .3 & 0 & 0 \\ 0 & .5 & 0 \end{bmatrix}$$

Suppose that a certain fraction of large fish are harvested before they reproduce. How large can that fraction be and still have the species survive?