Proofs for Practice Final.

P3 **a)** Let $F(x,y) = x^4 + y^3$. Then as $DF(x,y) = (4x^3, 3y^2)$ is never the zero vector unless $(x,y) = (0,0)$ which is not on the set considered here, we conclude this is a smooth manifold (see previous practice exam).

b) Using the previous part, we get $DF(2,-1) = (32,3)$ so the equation is $32\dot{x} + 3\dot{y}$.

P4 **a)** Let $f(x) = \log(1+x)$. Then, $f(0) = 0, f'(0) = 1, f''(0) = -\frac{1}{2}$, so the Taylor polynomial of degree two of f will be $P(h) = h - \frac{h^2}{2}$.

b) In this case, let $f(x,y) = \log(1 + x + xy = 2y)$, so $f(0,0) = 0$, $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 2$, $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$, $\frac{\partial^2 f}{\partial x^2}(0,0) = 0$, $\frac{\partial^2 f}{\partial y^2}(0,0) = -4$. Thus, the Taylor polynomial of degree two of f in this case will be $P(h_1,h_2) = 2h_2 + h_1 h_2 - 2h_2^2$.

P5 We have $xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$, but $\frac{x+y}{2}$ and $\frac{x-y}{2}$ are l.i. so the signature in this case is $(1,-1)$.

P6 Solving for $\nabla f = (0,0)$, we get:

$$4x^3 - 4y = 0$$

and

$$4y^3 - 4x = 0$$

so we obtain $x(x^8 - 1) = 0$. This yields, critical points are; $(0,0), (1,1)$ and $(-1,-1)$. Now $\frac{\partial^2 f}{\partial x^2}(x,y) = 12x^2$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 12y^2$, and $\frac{\partial^2 f}{\partial x \partial y}(x,y) = -4$. So at $(0,0)$ we have $det(Hess) = -16$. Hence $(0,0)$ is a saddle point. Now at both $(1,1)$ and $(-1,-1)$ $det(Hess) = 128$, and as $12 > 0$, we are looking at local minima.

P7 In this case,

$$\kappa = \frac{|e^x|}{(1+e^{2x})^{\frac{3}{2}}}.$$

Now note that as x goes to $-\infty$ the numerator e^x goes to zero. On the other hand as x goes to ∞ $\kappa \leq \frac{e^x}{e^{2 \cdot \frac{3}{2} x}} = e^{-2x}$ which goes to zero as well. Maximum curvature then is attained at a local maximum, in this case solving to find a critical point of κ we obtain $e^x - 3e^{3x} = 0$.

This implies $x = \frac{1}{2} \log \frac{1}{3}$. This turns out to give us a maximum, and thus the maximum curvature is

$$\frac{\left(\frac{1}{3}\right)^{\frac{1}{2}}}{(1 + \frac{1}{3})^{\frac{3}{2}}}.$$