**Practice Exam 3 - Covering sections 2.1-3.1**

1. Give the definition of

   (a) elementary matrices
   (b) span
   (c) linear independence
   (d) basis
   (e) dimension
   (f) kernel
   (g) image
   (h) rank
   (i) nullity
   (j) vector space
   (k) implicit function
   (l) graph
   (m) smooth manifold
   (n) parameterization of a manifold

2. State

   (a) The Dimension Formula
   (b) The Inverse Function Theorem (as presented in class)
   (c) The Implicit Function Theorem (as presented in class)

3. Consider the linear transformation from $\mathbb{R}^4 \to \mathbb{R}^3$ given by the matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 3 & -2 \\ 1 & 1 & -1 & 2 \end{bmatrix}$.

   (a) Find a basis for the kernel.
   (b) Find a basis for the image.
   (c) What are the rank and nullity for this transformation?
   (d) Is the vector $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in the image of this transformation?

4. Consider the subspace of $\mathbb{R}^3$ given by $x+y+2z=0$. One basis for this subspace is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$. A second basis for this subspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$. How do you write an arbitrary linear combination involving the second basis in terms of the first basis?

5. Consider the equation $y^2 + 2y + x^2 = 0$.

   (a) Use direct computation to determine where the equation defines $y$ implicitly as a function of $x$.
   (b) Verify $\det M \neq 0$ where $M = D_2 f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ (as required by the implicit function theorem as given in class) at these points.
   (c) Explain directly why the set of points $\begin{pmatrix} x \\ y \end{pmatrix}$ that satisfy the equation determines a smooth manifold.

(d) Explain why the set of points $\begin{pmatrix} x \\ y \end{pmatrix}$ determines a smooth manifold using Theorem 3.1.10 (a) which states: Let $M \subset \mathbb{R}^n$ be a subset, $U \subset \mathbb{R}^n$ be an open set, and $F : U \to \mathbb{R}^{n-k}$ be a $C^1$ mapping such that $M \bigcap U$ is the set of solutions to $F(\mathbf{z}) = \mathbf{0}$. If $[DF(\mathbf{z})]$ is onto for every $\mathbf{z} \in M \bigcap U$, then $M \bigcap U$ is a smooth $k$-dimensional manifold embedded in $\mathbb{R}^n$. If every $\mathbf{z} \in M$ is in such a $U$, then $M$ is a manifold. Give explicitly $n$, $k$, $U$, $[DF(z)]$, and explain why $[DF(z)]$ is onto for every $\mathbf{z} \in M \bigcap U$.

6. (From Spivak)

(a) If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f'(x) \neq 0$ for all $x \in \mathbb{R}$, show that $f$ is one-to-one. (Hint: You have had 2 homework problems that, combined, essentially give you this result - the intermediate value theorem for derivatives and the problem that if $f'(x) > 0$ on an interval, then $f$ is strictly increasing on that interval.)

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}$. Show that $\det \left[ Df \begin{pmatrix} x \\ y \end{pmatrix} \right] \neq 0$ for all $\begin{pmatrix} x \\ y \end{pmatrix}$ but $f$ is not one-to-one. Why does this not contradict the inverse function theorem?