Practice Exam 2

1. Give the definition of
   (a) An open set
   (b) A closed set
   (c) The interior of a set
   (d) The closure of a set
   (e) The boundary of a set
   (f) A compact set
   (g) The supremum and infinum of a function on a set
   (h) The maximum and minimum of a function on a set
   (i) The derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $\mathbf{a} \in \mathbb{R}^{\mathbf{n}}$.

2. State
   (a) The Mean Value Theorem
   (b) The Intermediate Value Theorem
   (c) The Intermediate Value Theorem for Derivatives
   (d) The Chain Rule

3. Do the following limits exist and, if so, what are they?
   (a) $\lim_{\binom{x}{y}\to\binom{0}{0}} \frac{\sqrt{|xy|}}{x^2+y^2}$
   (b) $\lim_{\binom{x}{y}\to\binom{0}{0}} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$
   (c) $\lim_{\binom{x}{y}\to\binom{0}{0}} \frac{\sqrt{|x|}y}{x^2+y^2}$
   (d) $\lim_{\binom{x}{y}\to\binom{0}{0}} \frac{xy}{\sqrt{x^2+y^2}}$

4. If $f(x) = x^2 \sin(1/x)$ differentiable at $x = 0$? Why or why not?

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y$. Prove that $\left[Df\begin{pmatrix} a \\ b \end{pmatrix}\right] = \begin{bmatrix} 2a & 1 \end{bmatrix}$.

6. Is $f(\vec{x}) = \vec{x}\sqrt{|\vec{x}|}$ differentiable at $\vec{x} = \vec{0}$? If so, what is its derivative and why?

7. What is the derivative (Jacobian) for the function $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} \sqrt{x^2+y^4+y^2z^2} \\ \sin^2(xyz) \end{bmatrix}$ at $\begin{pmatrix} \pi \\ 1 \\ 2 \end{pmatrix}$

8. Prove that for any bounded set which is not closed, there exists an unbounded function with that set as its domain and the real numbers as its co-domain.

9. Show that the function $f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{xy}{x^2+y^2}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ has partial derivatives at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ but is not differentiable there.

10. Find A^{-1} where
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$