Practice Problems for the Final Exam

Distribution	**Density**	**Where it lives**	**MGF**
Binomial	$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$k = 0, 1, \ldots, n$	$M(t) = (1 - p + pe^t)^n$
Geometric	$p(k) = p(1-p)^{k-1}$	$k = 1, 2, \ldots$	$M(t) = \dfrac{pe^t}{1-(1-p)e^t}$
Negative Binomial	$p(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$	$k = r, r+1, \ldots$	$M(t) = \left(\dfrac{pe^t}{1-(1-p)e^t}\right)^r$
Poisson	$p(k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$	$k = 0, 1, \ldots$	$M(t) = e^{\lambda(e^t - 1)}$
Normal	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$	$-\infty < x < \infty$	$M(t) = e^{\mu t} e^{\sigma^2 t^2/2}$
Gamma	$f(x) = \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$x > 0$	$M(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^\alpha$
Uniform	$f(x) = 1$	$0 < x < 1$	$M(t) = \dfrac{e^t - 1}{t}$

The exponential distribution is a special case of the Gamma distribution with $\alpha = 1$.

1. Argue that the sum of two independent Poisson random variables has to be Poisson with the sum of the two rates due to what Poisson random variables are and what they model.

2. Argue that the sum of two independent Poisson random variables has to be Poisson with the sum of the two rates by finding the density function of the sum from the density functions of the two components.

3. Argue that the sum of two independent Poisson random variables has to be Poisson with the sum of the two rates using moment generating functions. That is, find the moment generating function of a Poisson random variable and use the facts that the moment generating function for the sum is the product of the moment generating functions for the components and that moment generating functions determine the distribution when they exist in a neighborhood of zero.

4. Lie detector tests are notoriously flawed. Assume that a particular test gives false positives with probability 0.2 and false negatives with probability 0.1. In a large workforce assume that 15% of applicants lie. What is the probability that if the lie detector test says a random applicant lies that the applicant really is lying.

5. Let $X_n \to c$ in probability where c is a constant. Assume that g is a continuous function at c. Show that $g(X_n) \to g(c)$ in probability.

6. Let P be uniform on $[1/2, 1]$. Given $P = p$, let X be Bernoulli with parameter p. Find the conditional distribution of P given X.

7: A customer chooses the restaurant menu with probability 3/4 and the bar menu with probability 1/4. Meals on the restaurant menu cost an average of $20 with a standard deviation of $5 (variance 25). Meals on the bar menu cost an average of $12 with a standard deviation of $3 (variance 9).
(A) What is the mean cost of the meals this customer chooses?
(B) What is the variance and standard deviation of the cost of the meals this customer chooses? (You do not have to simplify or round the standard deviation; it is just the square root of the variance.)

8. Prove or find a concrete counterexample: If A and B are independent events, then A^c and B are independent events.

9. Use a counting argument to prove:

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

10. Give an example of two uncorrelated random variables which are not independent.

11. Use the delta method to find the approximate mean and variance of $Y = -\log(X)$ where X is uniform on $[0, 1]$.

12. Find the distribution of $Y = -\log(X)$ where X is uniform on $[0, 1]$ exactly and calculate its mean and variance exactly. Comment on whether you think the delta method is a good or bad approximation (this can be done without a calculator...)

13. Suppose you have 10 mailboxes and 20 identical letters.
(A) How many ways can you put the letters in the mailboxes?
(B) How many ways can you put the letters in the mailboxes so that no mailbox is empty?

14. Let U_1, U_2, ..., U_n be independent uniform $[0, 1]$ random variables. Let $U_{(n)}$ be the maximum.
(A) Find the density of $U_{(n)}$.
(B) Find the mean of $U_{(n)}$.
(C) Find the variance of $U_{(n)}$.
(D) Find the limiting distribution of the CDF of the standardized maximum, $U_{(n)}$.

15. (A) State Chebyshev's inequality.

(B) Let X be a binomial random variable with 18 trials and probability 1/3 of success. What lower bound on $\Pr(3 < X < 9)$ does Chebyshev's inequality imply?

16. Let P be uniform on $[0, 1]$. Given $P = p$, let X be a geometric random variable with parameter p. Find the conditional distribution of P given $X = k$. Hint: Set the problem up properly for half credit. The denominator can be calculated explicitly using integration by parts. Do that for full credit.

17. Suppose X_1 is normal with mean μ_1 and variance σ_1^2. Suppose X_2 is independent of X_1 and is normal with mean μ_2 and variance σ_2^2. Use moment generating functions to determine the distribution of $aX_1 + bX_2$.

18. Let X and Y have joint distribution $f(x, y) = e^{-y}$ for $0 < x < y < \infty$. Find the density function for the random variable $Z = X + Y$.

19. Determine the moment generating function for a uniform $[-1, 1]$ random variable.

20. Use a combinatorial argument to show that

$$\binom{n+1}{k+1} = \sum_{m=k}^{n} \binom{m}{k}$$

21. Argue that the sum of two independent Binomial random variables, one with n trials and one with m trials, but with the same probability of success, p, has to be Binomially distributed due to what Binomial random variables are and what they model.

22. Argue that the sum of two independent Binomial random variables with the same probability of success is Binomial finding the density function of the sum from the density functions of the two components.

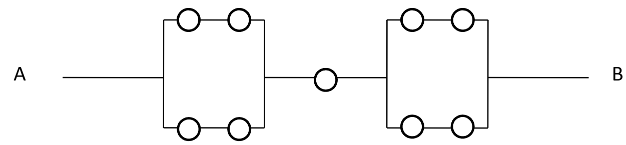
23. Argue that the sum of two independent Binomial random variables with the same probability of success has to be Binomial using moment generating functions. That is, find the moment generating function of a Binomial random variable and use the facts that the moment generating function for the sum is the product of the moment generating functions for the components and that moment generating functions determine the distribution when they exist in a neighborhood of zero.

24. Drug tests work differently than lie detector tests. For marijuana testing, a common test gives false positives with probability 0.02 approximately but false negatives with probability 0.40 approximately. Assume approximately 20% of high school seniors have used marijuana recently. What is the probability that a student is clean (has not used marijuana, at least not recently) given that the test for marijuana use is negative?

25. Suppose you have an urn with 1 red ball, 2 green balls, and 3 blue balls and you draw 3 balls out (without replacement.) Let X denote the number of red balls you drew and Y denote the number of green balls you drew.

(A) Find the joint distribution for X and Y.
(B) Find the conditional distribution for X given Y.
(C) Find the correlation between X and Y.

26. In the following diagram, circles represent switches that fail with probability p independent of each other. What is the probability that current flows from A to B?



27. Let X be an exponential random variable with rate $\lambda = 1$. Find the density of $Y = \ln X$.

28. Let $X_1, X_2, \ldots, X_n$ be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k. Let $X_{(1)}$ denote the minimum of the X_i's. Find the distribution of $X_{(1)}$.

29. Let $X_1, X_2, \ldots, X_n$ be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k. Let $X_{(n)}$ denote the maximum of the X_i's. Find the distribution of $X_{(n)}$.

30. Let $f(x) = c/x^3$ for $x > 1$ and let f be zero otherwise. For what value of c is f a probability density function? How would you use uniform $[0, 1]$ random numbers to generate random numbers from this distribution?