Homework Problem Set 11

(8.1.5) 1. Let X be a random variable with $E(X) = 0$ and $V(X) = 1$. What integer value k will assure us that $P(|X| \geq k) \leq .01$?

(Rice 5.7) 2. Show that if $X_n \to c$ in probability and if g is a continuous function, then $g(X_n) \to g(c)$ in probability.

(8.1.12) 3. (Chebyshev[1]) Assume that X_1, X_2, $\ldots$, X_n are independent random variables with possibly different distributions and let S_n be their sum. Let $m_k = E(X_k)$, $\sigma_k^2 = V(X_k)$, and $M_n = m_1 + m_2 + \cdots + m_n$. Assume that $\sigma_k^2 < R$ for all k. Prove that, for any $\epsilon > 0$,

$$P\left(\left|\frac{S_n}{n} - \frac{M_n}{n}\right| < \epsilon\right) \to 1$$

as $n \to \infty$.

(8.1.16) 4. In this exercise, we shall construct an example of a sequence of random variables that satisfies the weak law of large numbers, but not the strong law. The distribution of X_i will have to depend on i, because otherwise both laws would be satisfied. (This problem was communicated to us by David Maslen.)

Suppose we have an infinite sequence of mutually independent events $A_1, A_2, \ldots$. Let $a_i = P(A_i)$, and let r be a positive integer.

(a) Find an expression of the probability that none of the A_i with $i > r$ occur.

(b) Use the fact that $1 - x \leq e^{-x}$ to show that

$$P(\text{No } A_i \text{ with } i \geq r \text{ occurs}) \leq e^{-\sum_{i=r}^{\infty} a_i}$$

(c) (The second Borel-Cantelli lemma) Prove that if $\sum_{i=1}^{\infty} a_i$ diverges, then

$$P(\text{infinitely many } A_i \text{ occur}) = 1.$$

Now, let X_i be a sequence of mutually independent random variables such that for each positive integer $i \geq 2$,

$$P(X_i = i) = \frac{1}{2i \log i}, \quad P(X_i = -i) = \frac{1}{2i \log i}, \quad P(X_i = 0) = 1 - \frac{1}{i \log i}.$$

When $i = 1$ we let $X_i = 0$ with probability 1. As usual we let $S_n = X_1 + \cdots + X_n$. Note that the mean of each X_i is 0.

(d) Find the variance of S_n.

(e) Show that the sequence $\langle X_i \rangle$ satisfies the Weak Law of Large Numbers, i.e. prove that for any $\epsilon > 0$

$$P\left(\left|\frac{S_n}{n}\right| \geq \epsilon\right) \to 0\ ,$$

as n tends to infinity.

[1] P. L. Chebyshev, "On Mean Values," *J. Math. Pure. Appl.*, vol. 12 (1867), pp. 177–184.

We now show that $\{X_i\}$ does not satisfy the Strong Law of Large Numbers. Suppose that $S_n/n \to 0$. Then because

$$\frac{X_n}{n} = \frac{S_n}{n} - \frac{n-1}{n}\frac{S_{n-1}}{n-1} ,$$

we know that $X_n/n \to 0$. From the definition of limits, we conclude that the inequality $|X_i| \geq \frac{1}{2}i$ can only be true for finitely many i.

(f) Let A_i be the event $|X_i| \geq \frac{1}{2}i$. Find $P(A_i)$. Show that $\sum_{i=1}^{\infty} P(A_i)$ diverges (use the Integral Test).

(g) Prove that A_i occurs for infinitely many i.

(h) Prove that

$$P\left(\frac{S_n}{n} \to 0\right) = 0,$$

and hence that the Strong Law of Large Numbers fails for the sequence $\{X_i\}$.

(8.2.4) 5. Let X be a continuous random variable with values exponentially distributed over $[0, \infty)$ with parameter $\lambda = 0.1$.

(a) Find the mean and variance of X.

(b) Using Chebyshev's Inequality, find an upper bound for the following probabilities: $P(|X - 10| \geq 2)$, $P(|X - 10| \geq 5)$, $P(|X - 10| \geq 9)$, and $P(|X - 10| \geq 20)$.

(c) Calculate these probabilities exactly, and compare with the bounds in (b).