Homework Problem Set 10

(7.1.10) 1. (Lévy[1]) Assume that n is an integer, not prime. Show that you can find two distributions a and b on the nonnegative integers such that the convolution of a and b is the equiprobable distribution on the set 0, 1, 2, ..., $n-1$. If n is prime this is not possible, but the proof is not so easy. (Assume that neither a nor b is concentrated at 0.)

2. Use convolutions to find the distribution of the sum of two independent random variables that are both uniform on the integers $\{0, 1, 2, \ldots, n-1\}$.

3. Use convolutions to prove that the sum of two independent Poisson random variables, one with rate λ and one with rate μ, is itself Poisson with rate $\lambda + \mu$.

4. Use convolutions to prove that the sum of two independent Binomial random variables, each with success probability p, but one with n trials and the other with m trials, is $\text{Bin}(n+m, p)$.

5. Use convolutions to prove that the sum of two independent negative binomial random variables, each with the same success probability p, also has a negative binomial distribution.

6. Use convolutions to prove that the sum of two independent Gamma distributed random variables, each with the same rate λ, is also Gamma distributed with the sum of the two shapes as the shape of the sum.

7. Suppose that $R^2 = X^2 + Y^2$. Find f_{R^2} and f_R if X and Y are independent standard normal distributions.

(7.2.10) 8. Let X_1, X_2, ..., X_n be n independent random variables each of which has an exponential density with mean μ. Let M be the *minimum* value of the X_j. Show that the density for M is exponential with mean μ/n. *Hint*: Use cumulative distribution functions.

(7.2.13) 9. Particles are subject to collisions that cause them to split into two parts with each part a fraction of the parent. Suppose that this fraction is uniformly distributed between 0 and 1. Following a single particle through several splittings we obtain a fraction of the original particle $Z_n = X_1 \cdot X_2 \cdot \cdots \cdot X_n$ where each X_j is uniformly distributed between 0 and 1. Show that the density for the random variable Z_n is

$$f_n(z) = \frac{1}{(n-1)!}(-\log z)^{n-1}.$$

Hint: Show that $Y_k = -\log X_k$ is exponentially distributed. Use this to find the density function for $S_n = Y_1 + Y_2 + \cdots + Y_n$, and from this the cumulative distribution and density of $Z_n = e^{-S_n}$.

(7.2.14) 10. Assume that X_1 and X_2 are independent random variables, each having an exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density

$$f_Z(z) = (1/2)\lambda e^{-\lambda|z|} \ .$$

[1] See M. Krasner and B. Ranulae, “Sur une Proprieté des Polynomes de la Division du Circle”; and the following note by J. Hadamard, in *C. R. Acad. Sci.,* vol. 204 (1937), pp. 397–399.