Homework Problem Set 9

(6.3.3) 1. The lifetime, measure in hours, of the ACME super light bulb is a random variable T with density function $f_T(t) = \lambda^2 t e^{-\lambda t}$, where $\lambda = .05$. What is the expected lifetime of this light bulb? What is its variance?

(6.3.7) 2. Let X be a random variable with density function f_X. Show, using elementary calculus, that the function

$$\phi(a) = E((X-a)^2)$$

takes its minimum value when $a = \mu(X)$, and in that case $\phi(a) = \sigma^2(X)$.

(6.3.8) 3. Let X be a random variable with mean μ and variance σ^2. Let $Y = aX^2 + bX + c$. Find the expected value of Y.

(6.3.10) 4. Let X and Y be independent random variables with uniform density functions on $[0,1]$. Find

(a) $E(|X-Y|)$.

(b) $E(\max(X,Y))$.

(c) $E(\min(X,Y))$.

(d) $E(X^2+Y^2)$.

(e) $E((X+Y)^2)$.

(6.3.17) 5. Let X and Y be random variables. The *covariance* $\mathrm{Cov(X,Y)}$ is defined by (see Exercise 6.2.23)

$$\mathrm{cov(X,Y) = E((X-\mu(X))(Y-\mu(Y)))}\ .$$

(a) Show that $\mathrm{cov(X,Y) = E(XY) - E(X)E(Y)}$.

(b) Using (a), show that $\mathrm{cov}(X,Y) = 0$, if X and Y are independent. (Caution: the converse is *not* always true.)

(c) Show that $V(X+Y) = V(X) + V(Y) + 2\mathrm{cov}(X,Y)$.

(6.3.18) 6. Let X and Y be random variables with positive variance. The *correlation* of X and Y is defined as

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{V(X)V(Y)}}\ .$$

(a) Using Exercise 6.3.17(c), show that

$$0 \le V\left(\frac{X}{\sigma(X)} + \frac{Y}{\sigma(Y)}\right) = 2(1+\rho(X,Y))\ .$$

(b) Now show that

$$0 \le V\left(\frac{X}{\sigma(X)} - \frac{Y}{\sigma(Y)}\right) = 2(1-\rho(X,Y))\ .$$

(c) Using (a) and (b), show that

$$-1 \le \rho(X,Y) \le 1\ .$$

(6.3.19) 7. Let X and Y be independent random variables with uniform densities in $[0,1]$. Let $Z = X + Y$ and $W = X - Y$. Find

(a) $\rho(X,Y)$ (see Exercise 6.3.19).

(b) $\rho(X,Z)$.

(c) $\rho(Y,W)$.

(d) $\rho(Z,W)$.

(6.3.20) 8. When studying certain physiological data, such as heights of fathers and sons, it is often natural to assume that these data (e.g., the heights of the fathers and the heights of the sons) are described by random variables with normal densities. These random variables, however, are not independent but rather are correlated. For example, a two-dimensional standard normal density for correlated random variables has the form

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)} .$$

(a) Show that X and Y each have standard normal densities.

(b) Show that the correlation of X and Y (see Exercise 6.3.19) is ρ.

(6.3.21) 9. For correlated random variables X and Y it is natural to ask for the expected value for X given Y. For example, Galton calculated the expected value of the height of a son given the height of the father. He used this to show that tall men can be expected to have sons who are less tall on the average. Similarly, students who do very well on one exam can be expected to do less well on the next exam, and so forth. This is called *regression on the mean.* To define this conditional expected value, we first define a conditional density of X given $Y = y$ by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} ,$$

where $f_{X,Y}(x,y)$ is the joint density of X and Y, and f_Y is the density for Y. Then the conditional expected value of X given Y is

$$E(X|Y=y) = \int_a^b x f_{X|Y}(x|y)\, dx .$$

For the normal density in Exercise 6.3.20, show that the conditional density of $f_{X|Y}(x|y)$ is normal with mean ρy and variance $1-\rho^2$. From this we see that if X and Y are positively correlated $(0 < \rho < 1)$, and if $y > E(Y)$, then the expected value for X given $Y = y$ will be less than y (i.e., we have regression on the mean).

(6.3.28) 10. A long needle of length L much bigger than 1 is dropped on a grid with horizontal and vertical lines one unit apart. Show that the average number a of lines crossed is approximately

$$a = \frac{4L}{\pi} .$$