Homework Problem Set 6

(5.1.6) 1. Let $X_1,\ X_2,\ \ldots,\ X_n$ be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k. Let Y denote the minimum of the X_i's. Find the distribution of Y.

(5.1.7) 2. A die is rolled until the first time T that a six turns up.

(a) What is the probability distribution for T?

(b) Find $P(T > 3)$.

(c) Find $P(T > 6 | T > 3)$.

(5.1.10) 3. A census in the United States is an attempt to count everyone in the country. It is inevitable that many people are not counted. The U. S. Census Bureau proposed a way to estimate the number of people who were not counted by the latest census. Their proposal was as follows: In a given locality, let N denote the actual number of people who live there. Assume that the census counted n_1 people living in this area. Now, another census was taken in the locality, and n_2 people were counted. In addition, n_{12} people were counted both times.

(a) Given N, n_1, and n_2, let X denote the number of people counted both times. Find the probability that $X = k$, where k is a fixed positive integer between 0 and n_2.

(b) Now assume that $X = n_{12}$. Find the value of N which maximizes the expression in part (a). *Hint*: Consider the ratio of the expressions for successive values of N.

(5.1.11) 4. Suppose that X is a random variable which represents the number of calls coming in to a police station in a one-minute interval. In the text, we showed that X could be modelled using a Poisson distribution with parameter λ, where this parameter represents the average number of incoming calls per minute. Now suppose that Y is a random variable which represents the number of incoming calls in an interval of length t. Show that the distribution of Y is given by

$$P(Y = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}\ ,$$

i.e., Y is Poisson with parameter λt. *Hint*: Suppose a Martian were to observe the police station. Let us also assume that the basic time interval used on Mars is exactly t Earth minutes. Finally, we will assume that the Martian understands the derivation of the Poisson distribution in the text. What would she write down for the distribution of Y?

(5.1.23) 5. For a certain experiment, the Poisson distribution with parameter $\lambda = m$ has been assigned. Show that a most probable outcome for the experiment is the integer value k such that $m - 1 \le k \le m$. Under what conditions will there be two most probable values? *Hint*: Consider the ratio of successive probabilities.

(5.1.29) 6. The king's coinmaster boxes his coins 500 to a box and puts 1 counterfeit coin in each box. The king is suspicious, but, instead of testing all the coins in 1 box, he tests 1 coin chosen at random out of each of 500 boxes. What is the probability that he finds at least one fake? What is it if the king tests 2 coins from each of 250 boxes?

Number of deaths	Number of corps with x deaths in a given year
0	144
1	91
2	32
3	11
4	2

Table 1: Mule kicks.

(5.1.31) 7. In one of the first studies of the Poisson distribution, von Bortkiewicz[1] considered the frequency of deaths from kicks in the Prussian army corps. From the study of 14 corps over a 20-year period, he obtained the data shown in Table 1. Fit a Poisson distribution to this data and see if you think that the Poisson distribution is appropriate.

(5.1.34) 8. In the appeal of the *People v. Collins* case (see Exercise 4.1.28), the counsel for the defense argued as follows: Suppose, for example, there are 5,000,000 couples in the Los Angeles area and the probability that a randomly chosen couple fits the witnesses' description is 1/12,000,000. Then the probability that there are two such couples given that there is at least one is not at all small. Find this probability. (The California Supreme Court overturned the initial guilty verdict.)

(5.1.35) 9. A manufactured lot of brass turnbuckles has S items of which D are defective. A sample of s items is drawn without replacement. Let X be a random variable that gives the number of defective items in the sample. Let $p(d) = P(X = d)$.

(a) Show that

$$p(d) = \frac{\binom{D}{d}\binom{S-D}{s-d}}{\binom{S}{s}} .$$

Thus, X is hypergeometric.

(b) Prove the following identity, known as *Euler's formula*:

$$\sum_{d=0}^{\min(D,s)} \binom{D}{d}\binom{S-D}{s-d} = \binom{S}{s} .$$

(5.1.44) 10. Suppose that in the hypergeometric distribution, we let N and k tend to ∞ in such a way that the ratio k/N approaches a real number p between 0 and 1. Show that the hypergeometric distribution tends to the binomial distribution with parameters n and p.

[1] L. von Bortkiewicz, *Das Gesetz der Kleinen Zahlen* (Leipzig: Teubner, 1898), p. 24.