Homework Problem Set 8

(6.1.6) 1. A die is rolled twice. Let X denote the sum of the two numbers that turn up, and Y the difference of the numbers (specifically, the number on the first roll minus the number on the second). Show that $E(XY) = E(X)E(Y)$. Are X and Y independent?

(6.1.12) 2. Recall that in the *martingale doubling system* (see Exercise 1.1.10), the player doubles his bet each time he loses. Suppose that you are playing roulette in a *fair casino* where there are no 0's, and you bet on red each time. You then win with probability 1/2 each time. Assume that you enter the casino with 100 dollars, start with a 1-dollar bet and employ the martingale system. You stop as soon as you have won one bet, or in the unlikely event that black turns up six times in a row so that you are down 63 dollars and cannot make the required 64-dollar bet. Find your expected winnings under this system of play.

(6.1.21) 3. Let X be a random variable which is Poisson distributed with parameter λ. Show that $E(X) = \lambda$. *Hint*: Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

(6.1.29) 4. In the casino game of blackjack the dealer is dealt two cards, one face up and one face down, and each player is dealt two cards, both face down. If the dealer is showing an ace the player can look at his down cards and then make a bet called an *insurance* bet. (Expert players will recognize why it is called insurance.) If you make this bet you will win the bet if the dealer's second card is a *ten card*: namely, a ten, jack, queen, or king. If you win, you are paid twice your insurance bet; otherwise you lose this bet. Show that, if the only cards you can see are the dealer's ace and your two cards and if your cards are not ten cards, then the insurance bet is an unfavorable bet. Show, however, that if you are playing two hands simultaneously, and you have no ten cards, then it is a favorable bet. (Thorp[1] has shown that the game of blackjack is favorable to the player if he or she can keep good enough track of the cards that have been played.)

(6.1.31) 5. (Feller[2]) A large number, N, of people are subjected to a blood test. This can be administered in two ways: (1) Each person can be tested separately, in this case N test are required, (2) the blood samples of k persons can be pooled and analyzed together. If this test is *negative,* this one test suffices for the k people. If the test is *positive,* each of the k persons must be tested separately, and in all, $k + 1$ tests are required for the k people. Assume that the probability p that a test is positive is the same for all people and that these events are independent.

(a) Find the probability that the test for a pooled sample of k people will be positive.

(b) What is the expected value of the number X of tests necessary under plan (2)? (Assume that N is divisible by k.)

(c) For small p, show that the value of k which will minimize the expected number of tests under the second plan is approximately $1/\sqrt{p}$.

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[1]E. Thorp, *Beat the Dealer* (New York: Random House, 1962).

[2]W. Feller, *Introduction to Probability Theory and Its Applications,* 3rd ed., vol. 1 (New York: John Wiley and Sons, 1968), p. 240.

(6.1.35) 6. A coin is tossed until the first time a head turns up. If this occurs on the nth toss and n is odd you win $2^n/n$, but if n is even then you lose $2^n/n$. Then if your expected winnings exist they are given by the convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

called the alternating *harmonic series.* It is tempting to say that this should be the expected value of the experiment. Show that if we were to do this, the expected value of an experiment would depend upon the order in which the outcomes are listed.

(6.2.4) 7. X is a random variable with $E(X) = 100$ and $V(X) = 15$. Find

(a) $E(X^2)$.

(b) $E(3X + 10)$.

(c) $E(-X)$.

(d) $V(-X)$.

(e) $D(-X)$.

(6.2.11) 8. A number is chosen at random from the integers 1, 2, 3, ..., n. Let X be the number chosen. Show that $E(X) = (n + 1)/2$ and $V(X) = (n - 1)(n + 1)/12$. *Hint*: The following identity may be useful:

$$1^2 + 2^2 + \cdots + n^2 = \frac{(n)(n+1)(2n+1)}{6}\ .$$

(6.2.23) 9. If X and Y are any two random variables, then the *covariance* of X and Y is defined by $\mathrm{Cov}(X, Y) = E((X - E(X))(Y - E(Y)))$. Note that $\mathrm{Cov}(X, X) = V(X)$. Show that, if X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$; and show, by an example, that we can have $\mathrm{Cov}(X, Y) = 0$ and X and Y not independent.

(6.2.29) 10. Let X be Poisson distributed with parameter λ. Show that $V(X) = \lambda$.